

Visualizing Fractions

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Abstract:

This article is a presentation of the file “Visualizing Fractions” constructed in Ti-Nspire CAS. It will in some detail explain the construction itself, what it contains and the pedagogical thinking that is the basis for the construction. “Visualizing Fractions” deals with the most common forms of operations on fractions, addition, subtraction, multiplication and division. By the use of sliders, two independent fractions can be determined and viewed in pie form, along with the result. The type of operation can be chosen by a slider and the result will change accordingly. One important focus of the file is to visually explain why we get the result we get. For every choice of operation, a possibility to verify the answer visually will be offered, a different approach for each of the four operations. The article contains a link to the file. “Visualizing Fractions” is freeware and can be downloaded and used with Ti-Nspire™ CAS, Ti-Nspire™ CAS App for iPad® or used with Ti-Nspire™ Document Player.

Introduction

Fractions and fractional operations are a challenge to understand and master. Teachers everywhere do their utmost to make their students understand the concept of a fraction and the rules that need to be applied when performing fractional operations. Despite all this effort, all aspects of fractions still remain difficult for some, even when students grow older. Attempts to visualize fractions are made by most teachers regularly. Whether the teacher pictures the fractions as pizzas or rectangles or other forms in his or her attempt, the notion that teachers think it is important to visualize is fundamental. This is the reason for the construction of “Visualizing Fractions”.

In the following I will explain in some detail how it has been carried out and the mathematical didactics I have used in the design. I will address each issue in the following order:

1. The design
2. The construction
3. The didactics
4. The conclusion

1. The design

I have chosen to use circles and circle segments. Ti-Nspire CAS [6] comes both as software and as a calculator. I have made the full version in the software and a moderated version that can be used on the calculator Ti-Nspire CX CAS [7]. The working area of the file is a Graph page that includes two fractions, one yellow and one blue. The answer in green is the third fraction in the page. Since this is not a calculator as such, but rather a pedagogical tool, limitations have been made to how big the fractions can be. The two independent fractions cannot exceed 2, and the answer cannot exceed 4.

I have also included variables $n1$, $d1$, $n2$, $d2$, k , cd , $ex1$, $ex2$, l_r and $check$, all controlled by sliders. I will explain below the purpose of these variables. Finally, an information area to explain what to do has been placed strategically in good view.

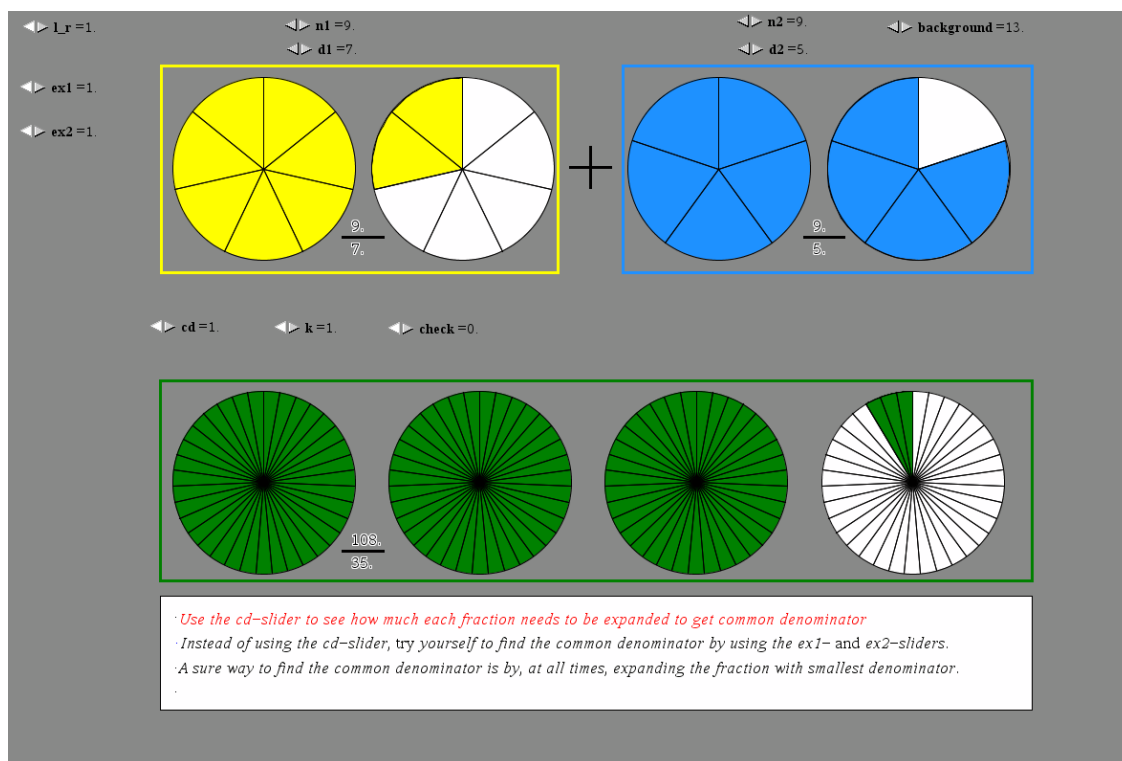


Figure 1. The three fractions of the Graph page.

2. The construction

Step 1 in the construction is to define variables that will allow us to change the fractions as we please. In the yellow fraction, $n1$ is the numerator and $d1$ is the denominator. Both are controlled by sliders so the fraction can be changed just by using these. For the blue fraction, $n2$ and $d2$ are similarly numerator and denominator.

Drawing circle segments is not integrated in Ti-Nspire CAS. So for me to be able to draw circle segments anyway they have to be constructed. It is a special and very useful feature in Ti-Nspire™ CAS [1] that polygons can be attached to scatter plots. So I use a combination of sequences generated in a List & Spreadsheet, scatter plots of these and the polygon tool in the Graph page to create a fraction. In Figure 2 the red plot is used to create the yellow polygon and the black connected scatter plot defines the denominator. The image of the fraction also includes a circle with radius rad .

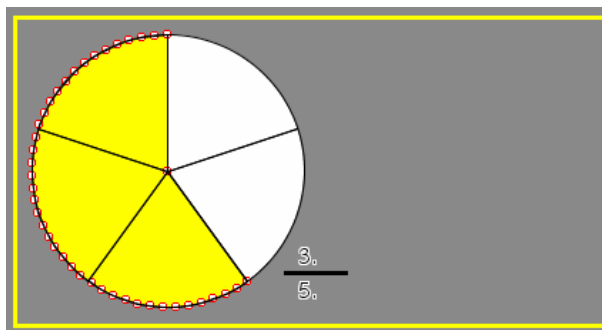


Figure 2. The scatter plot that builds the polygon

The x-values and y-values of the scatter plots are generated by the following formulas:

$$\text{seq}\left(\text{when}\left(-0.1 < \text{mod}(a,2) < 0.1, \text{rad} \cdot \cos\left(\frac{0.5 \cdot a}{d11} \cdot 2 \cdot \pi + \frac{\pi}{2}\right) + a1, a1\right), a, 0, 2, d11-1\right)$$

$$\text{seq}\left(\text{when}\left(-0.1 < \text{mod}(a,2) < 0.1, \text{rad} \cdot \sin\left(\frac{0.5 \cdot a}{d11} \cdot 2 \cdot \pi + \frac{\pi}{2}\right) + b1, b1\right), a, 0, 2, d11-1\right)$$

Figure 3. The formulas for the scatter plot that defines the denominator

$$\text{when}\left(\nu I \leq 2 \cdot \pi, \text{augment}\left(\{a1\}, \text{seq}\left(\text{rad} \cdot \cos\left(\frac{a}{n} \cdot \nu I + \frac{\pi}{2}\right) + a1, a, 0, n\right)\right), \text{when}\left(2 \cdot \pi < \nu I \leq 4 \cdot \pi, \text{augment}\left(\{a2\}, \text{seq}\left(\text{rad} \cdot \cos\left(\frac{a}{n} \cdot (\nu I - 2 \cdot \pi) + \frac{\pi}{2}\right) + a2, a, 0, n\right)\right), \text{seq}(_, a, 0, 0)\right)$$

$$\text{when}\left(\nu I \leq 2 \cdot \pi, \text{augment}\left(\{b1\}, \text{seq}\left(\text{rad} \cdot \sin\left(\frac{a}{n} \cdot \nu I + \frac{\pi}{2}\right) + b1, a, 0, n\right)\right), \text{when}\left(2 \cdot \pi < \nu I \leq 4 \cdot \pi, \text{augment}\left(\{b2\}, \text{seq}\left(\text{rad} \cdot \sin\left(\frac{a}{n} \cdot (\nu I - 2 \cdot \pi) + \frac{\pi}{2}\right) + b2, a, 0, n\right)\right), \text{seq}(_, a, 0, 0)\right)$$

Figure 4. The formulas for the scatter plot in Figure 2

Every fraction is in a one to one relation with an angle so νI is defined as $\frac{n1}{d1} \cdot 2\pi$. The formulas in Figure 3 will produce values on a circle with radius equal to rad and every other value in the sequence will be the center point $(a1, b1)$ defined as a point in the Graph page. The purpose of this is that when the points of the scatter plot are connected they form a star shaped plot that divides the circle into $d1$ parts. Figure 4 shows the formulas for the red scatter plot seen in Figure 2.

The variable k will change the type of operation we want to perform. $k=1$ is addition, $k=2$ is subtraction, $k=3$ is multiplication and $k=4$ is division. The variable cd will have as effect that for $k=1$ and $k=2$ the fractions are changed to have common denominator. For $k=3$ and $k=4$ it renders the fractions unreduced.

In the construction of the green fraction, the answer of the calculation, we use the following formulas defined in a Notes page:

$$f(k) := \begin{cases} \frac{n1 \cdot d2 + n2 \cdot d1}{\text{gcd}(n1 \cdot d2 + n2 \cdot d1, d1 \cdot d2)}, & 0.5 < k < 1.5 \\ \frac{n1 \cdot d2 - n2 \cdot d1}{\text{gcd}(n1 \cdot d2 - n2 \cdot d1, d1 \cdot d2)}, & 1.5 < k < 2.5 \\ \frac{n1 \cdot n2}{\text{gcd}(n1 \cdot n2, d1 \cdot d2)}, & 2.5 < k < 3.5 \\ \frac{n1 \cdot d2}{\text{gcd}(n1 \cdot d2, d1 \cdot n2)}, & 3.5 < k < 4.5 \end{cases} \rightarrow \text{Done } g(k) := \begin{cases} \frac{d1 \cdot d2}{\text{gcd}(n1 \cdot d2 + n2 \cdot d1, d1 \cdot d2)}, & 0.5 < k < 1.5 \\ \frac{d1 \cdot d2}{\text{gcd}(n1 \cdot d2 - n2 \cdot d1, d1 \cdot d2)}, & 1.5 < k < 2.5 \\ \frac{d1 \cdot d2}{\text{gcd}(d1 \cdot d2, n1 \cdot n2)}, & 2.5 < k < 3.5 \\ \frac{d1 \cdot n2}{\text{gcd}(d1 \cdot n2, n1 \cdot d2)}, & 3.5 < k < 4.5 \end{cases} \rightarrow \text{Done}$$

Figure 5. Formulas for the numerator and denominator of the green fraction

So for different values of k , $f(k)$ is the numerator and $g(k)$ is the denominator. These are defined as $n3$ and $d3$ respectively. In order to understand what is actually happening, the following formulas for numerators and denominators are needed.

$$\begin{aligned}
 n11 &:= \text{when} \left(0.5 < cd < 1.5, n1 \cdot ex1, \text{when} \left(1.5 < cd < 2.5 \text{ and } 0.5 < k < 2.5, \frac{n1}{d1} \cdot \text{lcm}(d1, d2), n1 \right) \right) \rightarrow 1. \\
 d11 &:= \text{when} \left(0.5 < cd < 1.5, d1 \cdot ex1, \text{when} \left(1.5 < cd < 2.5 \text{ and } 0.5 < k < 2.5, \text{lcm}(d1, d2), d1 \right) \right) \rightarrow 3. \\
 n22 &:= \text{when} \left(0.5 < cd < 1.5, n2 \cdot ex2, \text{when} \left(1.5 < cd < 2.5 \text{ and } 0.5 < k < 2.5, \frac{n2}{d2} \cdot \text{lcm}(d1, d2), n2 \right) \right) \rightarrow 1. \\
 d22 &:= \text{when} \left(0.5 < cd < 1.5, d2 \cdot ex2, \text{when} \left(1.5 < cd < 2.5 \text{ and } 0.5 < k < 2.5, \text{lcm}(d1, d2), d2 \right) \right) \rightarrow 7. \\
 n33 &:= \text{when} \left(0.5 < cd < 1.5, n3 \cdot ex3, \text{when} \left(1.5 < cd < 2.5 \text{ and } 0.5 < k < 1.5, \left(\frac{n1}{d1} + \frac{n2}{d2} \right) \cdot \text{lcm}(d1, d2), \text{when} \left(1.5 < cd < 2.5 \right. \right. \right. \\
 &\quad \left. \left. \left. \text{and } 1.5 < k < 2.5, \left(\frac{n1}{d1} - \frac{n2}{d2} \right) \cdot \text{lcm}(d1, d2), \text{when} \left(1.5 < cd < 2.5 \text{ and } 2.5 < k < 3.5, n1 \cdot n2, n1 \cdot d2 \right) \right) \right) \right) \rightarrow 10. \\
 d33 &:= \text{when} \left(0.5 < cd < 1.5, d3 \cdot ex3, \text{when} \left(1.5 < cd < 2.5 \text{ and } 0.5 < k < 2.5, \text{lcm}(d1, d2), \text{when} \left(1.5 < cd < 2.5 \text{ and } 2.5 < k < 3.5, d1 \cdot d2, d1 \cdot n2 \right) \right) \right) \rightarrow 21.
 \end{aligned}$$

Figure 6. The formulas for the numerators and denominators when the variables k and cd are used.

The purpose of these formulas is that they will change the partitioning of the fractions according to what we want to study. For example, if you study the formula in Figure 3 more closely, you will see that the variable $d11$ is used there instead of $d1$.

3. The didactics

Addition

When we add two or more fractions, it is my intention to show that in order to do it correctly, the fractions have to be expanded to have common denominator. That is possible to do in two ways. The easiest way is to change the cd -slider from 1 to 2 in which case the formulas in Figure 6 make sure that the fractions are expanded to have common denominator. The numerator is multiplied by $\text{lcm}(d1, d2)$ (the least common multiple of $d1$ and $d2$) and divided by $d1$. Similarly the denominator is just changed to $\text{lcm}(d1, d2)$. The other way is to use the $ex1$ - and $ex2$ -sliders to manually expand each fraction a little by little. If you constantly expand the fraction with the smallest denominator, you will eventually reach a point when you have common denominators. You will be told in the information area when you have succeeded.

In Figure 8 below it is possible to see that in order for us to calculate $\frac{4}{5} + \frac{2}{3}$ we have to expand the first fraction by 3 and the second fraction by 5. The common denominator is 15.

As a verification of the answer it is now possible to push the $check$ -slider. The green of the answer fraction is split up in two parts, one yellow and one blue. This is made possible by the construction of two new polygons with scatter plots as earlier. Also two polygons (purple colored in Figure 9) will count each part of the answer fraction as you continue to push the $check$ -slider.

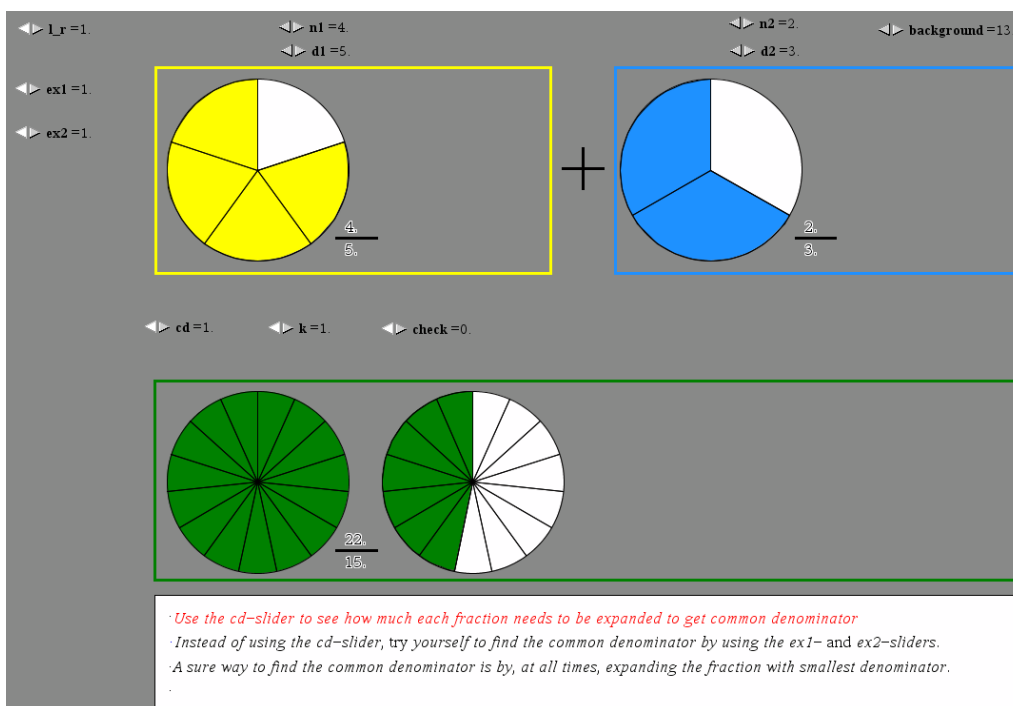


Figure 7: Initial stage of addition

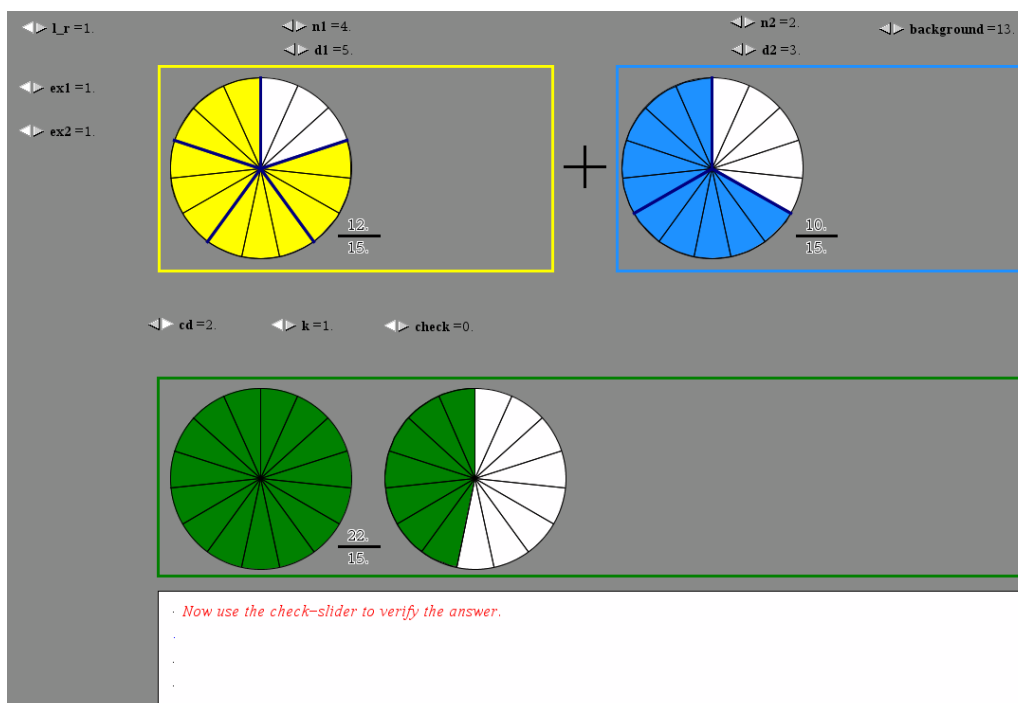


Figure 8. The fractions have been expanded to have common denominator

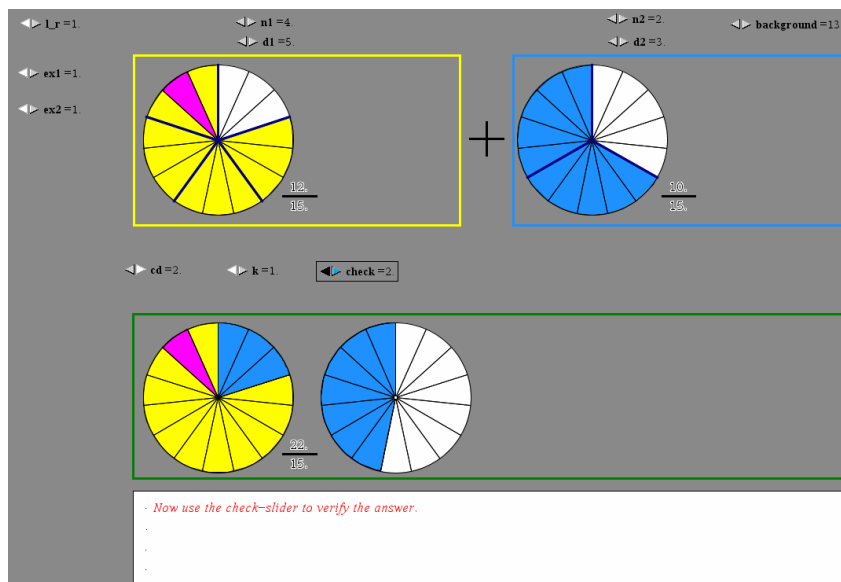


Figure 9. Verification of the answer

The first sequence in Figure 10 is the x-values for the yellow polygon in the answer, while the second and longer sequence formula is the purple moveable polygon in the answer. Similar sequences are of course needed for the y-values and the remaining of the four polygons.

$$\text{when}\left(\text{check}>0 \text{ and } 0.5<k<1.5 \text{ and } 1.5<cd<2.5 \text{ and } \sqrt{1}\leq 2: \pi,\text{augment}\left(\{a5\},\text{seq}\left(\text{rad}\cdot\cos\left(\frac{a}{n}\cdot\sqrt{1}+\frac{\pi}{2}\right)+a5,a,0,n\right)\right),\text{when}\left(\text{check}>0 \text{ and } 0.5<k<1.5 \text{ and } 1.5<cd<2.5\right.\right.$$

$$\left.\left.\text{and } 2\cdot\pi<\sqrt{1}\leq 4: \pi,\text{augment}\left(\{a6\},\text{seq}\left(\text{rad}\cdot\cos\left(\frac{a}{n}\cdot(\sqrt{1}-2)\cdot\pi+\frac{\pi}{2}\right)+a6,a,0,n\right)\right),\text{seq}(_,a,0,0)\right)\right)$$

$$\text{when}\left(0.5<k<2.5 \text{ and } 1.5<cd<2.5 \text{ and } 0.5<\text{check}\leq|n33|+\frac{1}{10000} \text{ and } \frac{(\text{check}-1)\cdot 2\cdot\pi}{d33}<\sqrt{v},\text{augment}\left(\{a5\},\text{seq}\left(\text{rad}\cdot\cos\left(\frac{a}{20}\cdot\frac{2\cdot\pi}{d33}+\frac{(\text{check}-1)\cdot 2\cdot\pi}{d33}+\frac{\pi}{2}\right)+a5,a,0,20\right)\right),\text{when}\left(0.5<k<2.5 \text{ and } 1.5<cd<2.5 \text{ and } 0.5<\text{check}\leq|n33|+\frac{1}{10000}\right.\right.$$

$$\left.\left.\text{and } \sqrt{v}\leq\frac{(\text{check}-1)\cdot 2\cdot\pi}{d33}<2\cdot\sqrt{v},\text{augment}\left(\{a6\},\text{seq}\left(\text{rad}\cdot\cos\left(\frac{a}{20}\cdot\frac{2\cdot\pi}{d33}+\frac{(\text{check}-1)\cdot 2\cdot\pi}{d33}+\frac{\pi}{2}\right)+a6,a,0,20\right)\right),\text{when}\left(0.5<k<2.5 \text{ and } 1.5<cd<2.5\right.\right.$$

$$\left.\left.\text{and } 2\cdot\sqrt{v}\leq\frac{(\text{check}-1)\cdot 2\cdot\pi}{d33}<3\cdot\sqrt{v},\text{augment}\left(\{a7\},\text{seq}\left(\text{rad}\cdot\cos\left(\frac{a}{20}\cdot\frac{2\cdot\pi}{d33}+\frac{(\text{check}-1)\cdot 2\cdot\pi}{d33}+\frac{\pi}{2}\right)+a7,a,0,20\right)\right),\text{when}\left(0.5<k<2.5 \text{ and } 1.5<cd<2.5\right.\right.$$

$$\left.\left.\text{and } 0.5<\text{check}\leq|n33|+\frac{1}{10000} \text{ and } 3\cdot\sqrt{v}\leq\frac{(\text{check}-1)\cdot 2\cdot\pi}{d33}<4\cdot\sqrt{v},\text{augment}\left(\{a8\},\text{seq}\left(\text{rad}\cdot\cos\left(\frac{a}{20}\cdot\frac{2\cdot\pi}{d33}+\frac{(\text{check}-1)\cdot 2\cdot\pi}{d33}+\frac{\pi}{2}\right)+a8,a,0,20\right)\right),\text{seq}(_,a,0,0)\right)\right)\right)$$

Figure 10. Formulas for verification of answer

Rather long formulas, but they are needed for all eventualities.

Subtraction

Most of what has been written about addition also applies to subtraction. However, when it comes to the verification of the answer things have to be done differently. Changing the *check*-slider away from -1 will move either the blue fraction over to the yellow one or vice versa, depending on which is bigger. At the same time the green fraction changes color either to yellow or blue, again depending on whether the answer is positive or negative (see Figures 12 and 13). It is a special feature of Ti-Nspire CAS that the color of objects and text is defined by a number ranging from 1 to 15, so by defining a formula that decides which color in which context it is

possible to do this. Figure 11 gives the formula I have used to color the green polygon either yellow or blue.

`pol1:=when(1.5<k<2.5 and v1≥v2 and 1.5<cd<2.5 and check>0.5,9,when(1.5<k<2.5 and v1<v2 and 1.5<cd<2.5 and check>0.5,3,when(0.5<k<1.5 and 1.5<cd<2.5 and check>0.5,3,4)))`

Figure 11. Formula for changing color of the answer

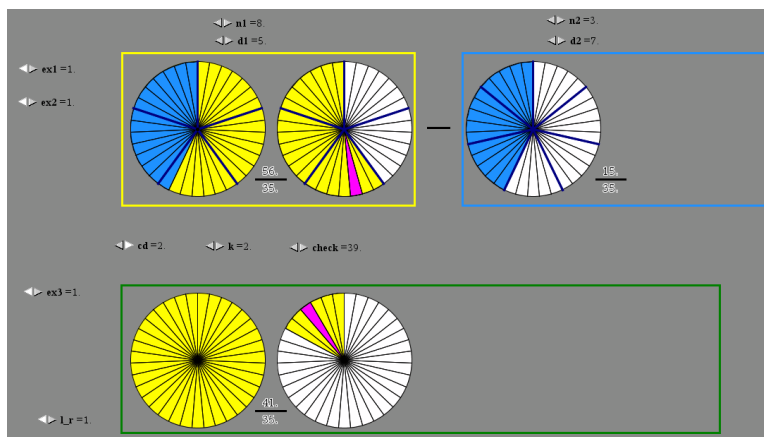


Figure 12: Verification of subtraction when yellow fraction is bigger than blue fraction

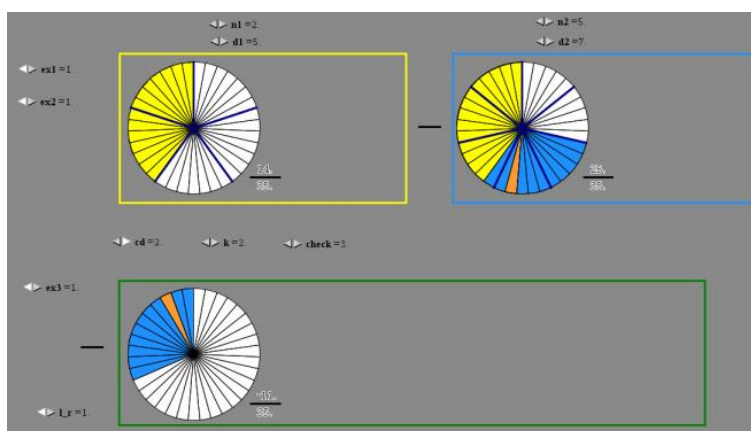


Figure 13. Verification of subtraction when yellow fraction is smaller than blue fraction

As with the idea of changing color, Ti-Nspire™ CAS [6] can also let us view text or objects according to certain conditions, a feature I use in several contexts. What kind of information you receive in the information area, depends on what kind of operation you are performing. As an example, some text is visible when you add, hidden when you multiply.

Multiplication

The rule for multiplication of fractions is straightforward enough, numerator multiplied by numerator and denominator multiplied by denominator. By pushing the *cd*-slider to 2, you can verify that it is so. Still, many students do not actually understand or have an intuitive understanding what the answer should be. Every fraction is in fact a percentage, so multiplying by a fraction is exactly the same as taking a certain percentage of something. One of the real benefits of seeing the fractions as we do in “Visualizing Fractions”, is when we work with multiplication and division. To visually experience how the answer gets smaller when we increase the denominator and how the answer gets bigger when we decrease the denominator is enlightening. Multiplication is commutative so in my attempt to verify the answer I have introduced the variable *l_r* (left-right).

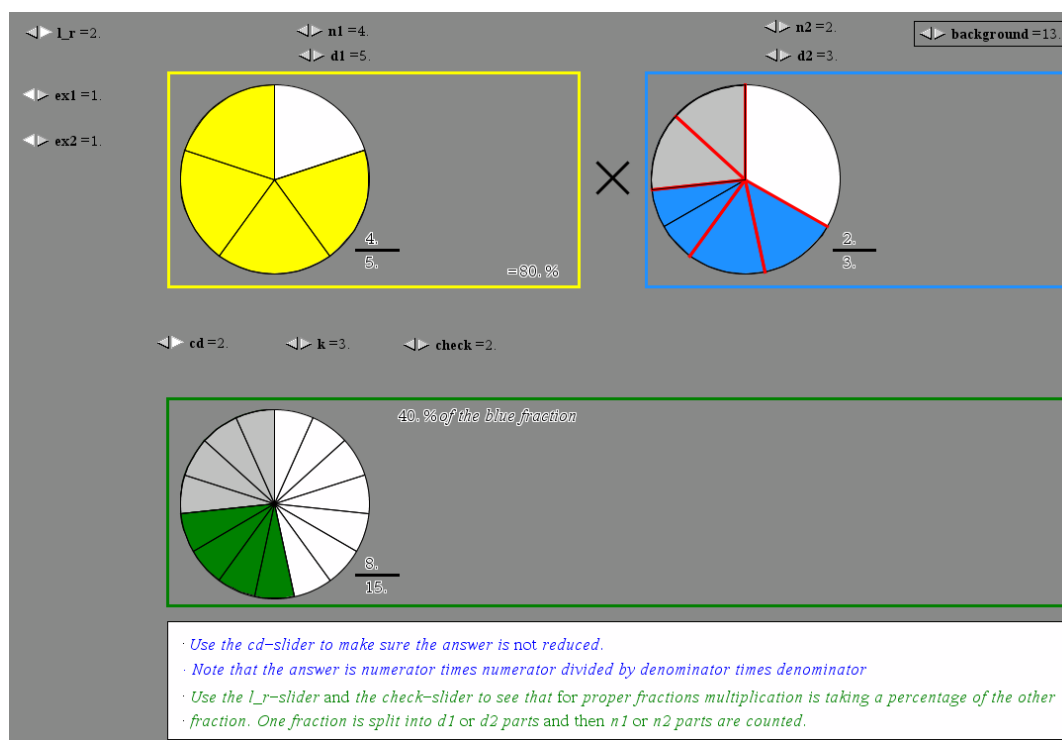


Figure 14. Blue fraction has been split up into *d1* parts

In Figure 14 *l_r* is 2 and the blue fraction has been split into *d1*=5 parts and two of these parts have been counted. To get the answer we need to continue to push the check-slider until we have reached 4 of the 5 parts or (80%) of the blue fraction.

We will get exactly the same answer if we split the yellow fraction into *d2*=3 parts and count 2 of these (67.7%) as can be seen in Figure 15. The variable *l_r* is now 1.

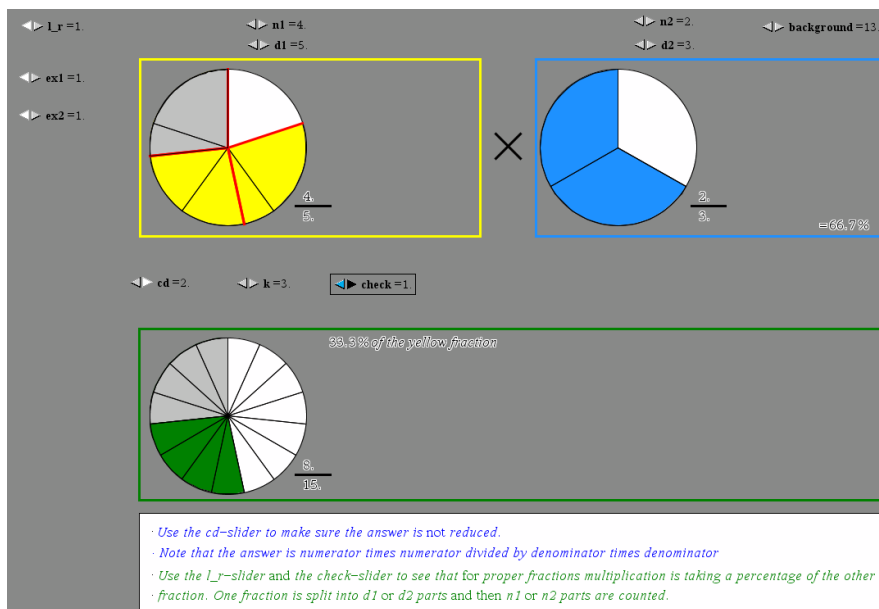


Figure 15. Yellow fraction has been split up into $d2$ parts

Division

Even more so than with multiplication, the possibility to actually see what happens when you divide by a fraction is an awakening. The smaller fraction you divide by, the bigger the answer will be. Change the blue fraction either by increasing the denominator or by decreasing the numerator and see the effect on the answer. As verification of the answer the same principle as for multiplication has been used. Push the check-slider and you will see that the yellow fraction is now split up into $n2$ parts. Continue to push the check-slider and you will count up $d2$ such parts.

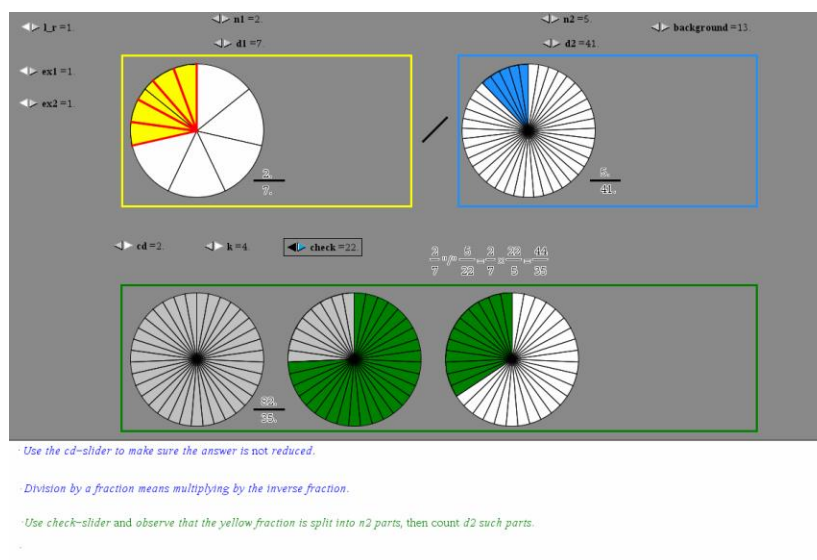


Figure 16. Yellow fraction has been split up into $n2$ parts

4. The conclusion

Understanding fractions and fractional operations is an important part of mathematics curricula in countries all over the world. To make pupils understand this vital part of mathematics, teachers have always used visualization in some form. It is well documented that using visualization, as a teaching method, is effective, regardless of the student's learning style. Fishbein [2] states that by adding an image to one's explanations, students understand more. Arcavi [1] gives a definition of what it means to visualize: "Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings". Lee and Ferrucci [3] found that the use of virtual manipulatives, in their own words, "*enhances the students' thinking and creativity*". D. Bezáková [4], in her study of students using Logotron Visual Fractions [5], observed that "*visualization of fractions by many different representations enables students to create better comprehension of fractions*". "Visualizing Fractions" can be downloaded as a tns-file from my home page www.oysteinnordvik.no or using the links below for direct access to them. The file can be used with Ti-Nspire™ CAS [6] or used with Ti-Nspire™ Document Player [8] or used with Ti-Nspire™ CAS App for iPad® [9]. It is therefore not necessary to install Ti-Nspire™ CAS [6] on your computer in order to use "Visualizing Fractions".

The software version and the handheld version can be downloaded from here:

[Visualizing Fractions SW](#)

[Visualizing Fractions HH](#)

Acknowledgements

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References

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http://stwww.weizmann.ac.il/departement40/publications/publications_arcavi.html
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- [3] Ngan Hoe Lee & Beverly J. Ferrucci (2012). Enhancing Learning Of Fraction Through The Use Of Virtual Manipulatives. The Electronic Journal of Mathematics and Technology, Volume 6, Number 2, ISSN 1933-2823
- [4] Bezáková, D. 2008. Visual fractions evaluation. In Research, Reflection and Innovations in Integrating ICT in Education, Vol. 3. Badajoz: Formatex, 2009, p.1493-1497.
- [5] Logotron Visual Fractions, educational software, © Logotron Ltd.

Software Packages:

- [6] TI-Nspire™ CAS Teacher Version, version 3.6.0.546, a product from Texas Instruments.
- [7] Ti-Nspire™ CX CAS Handheld, version 3.6.0.546, a product from Texas Instruments.
- [8] Ti-Nspire™ Document Player, Powered by Ti-Nspire™ Technology, <http://education.ti.com/en/us/document-player>
- [9] Ti-Nspire™ CAS App for iPad®